Strange solutions to field theories in one spatial dimension

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Many models of interacting particles rely heavily for their solution on restriction to one-dimensional motion and a linearized kinetic energy. We examine this in detail, and find that the linearization can lead to patently strange and possible spurious solutions in first quantization. The usual, correct solutions are obtained only in second quantization. The strange solutions do not reduce to the usual plane wave determinantal solutions, even when the interactions are extinguished, and have the character of a condensed phase — a sort of Wigner lattice—for arbitrary interactions.

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We examine a class of rather simple fermion Hamiltonians in one spatial dimension. These are exactly soluble, in the sense that one can exhibit explicitly their "correct," physically acceptable solutions. We also discover a totally new class of solutions for which no simple physical interpretation exists, which we label "strange." It has been very popular recently to linearize the kinetic energy, so as to obtain easy and convenient solutions of difficult problems. We shall examine this premise. Let us start with the simplest case, a one-component theory for N fermions characterized by the following Hamiltonian:

$$\begin{aligned} H &= -i\int_0^L dx\ \psi^\dagger(x)\partial_x\psi(x) \\ &\quad + \lambda_1 \int dx \int dx' \psi^\dagger(x)\,\psi(x)\,V(x-x')\psi^\dagger(x')\psi(x') \\ &= \mathrm{KE} + \mathrm{PE}. \end{aligned} \tag{1}$$

Here $\psi(x)$ is a fermion field operator:

$$\{\psi(x), \psi^\dagger(x')\} = \delta(x-x'), \quad \{\psi(x), \psi(x')\} = 0. \tag{2}$$

In terms of momentum operators c_k, ψ may be written

$$\psi(x) = L^{-1/2}\sum_k e^{ikx} c_k. \tag{3}$$

We also define the particle-density (or "current") operators:

$$\rho_q = \sum_k c_{k+q}^\dagger c_k, \quad \rho_{-q} = \rho_q^\dagger, \quad \text{and } N = \rho_0. \tag{4}$$

While the c_k's anticommute ($\{c_k, c_{k'}^\dagger\} = \delta_{k,k'}$), the ρ_q's commute:

$$[\rho_q, \rho_{q'}] = 0, \quad \text{all } q,q', \tag{5a}$$

as long as the number N is finite. If however, the Fermi–Dirac sea is filled, thus $N = \infty$, the ρ_q's satisfy a different set of commutation relations[1]:

$$[\rho_q, \rho_{q'}] = (qL/2\pi)\,\delta_{q,q'}. \tag{5b}$$

[It is very easy to verify (5b) when both sides of the equation operate on the Fermi–Dirac sea, the state in which all $k < k_F$ are occupied and all $k > k_F$ are unoccupied. The proof of (5b) as an operator identity is given elsewhere.[1]] In connection with (5b) we define a set of Bose operators a_q:

$$\rho_q = (|q|L/2\pi)^{1/2} \times \begin{cases} a_{|q} & \text{for } q < 0 \\ a_q^\dagger & \text{for } q > 0 \end{cases} \tag{6}$$

the a_q's— now defined only for $q > 0$— satisfy the standard Bose–Einstein commutation relations:

$$[a_q, a_{q'}] = 0, \quad [a_q, a_{q'}^\dagger] = \delta_{q,q'}. \tag{7}$$

Before proceeding to the exact solution of (1), we require one more algebraic identity, concerning the KE, which takes on the following form:

$$-i\int dx\ \psi^\dagger(x)\,\partial_x\psi(x) = \sum k c_k^\dagger c_k. \tag{8}$$

After subtraction of the energy of the Fermi–Dirac sea, it can be written as[2]

$$\sum k c_k^\dagger c_k - \sum_{k<k_F} k = \frac{2\pi}{L}\sum_{q>0}\rho_q^\dagger \rho_q = \sum_{q>0} q a_q^\dagger a_q. \tag{9}$$

The normal-mode operators a_q were, in fact, first introduced by Tomonaga.[3] It is seen that the free-fermion kinetic energy can be written as the energy of the decoupled normal modes. The solution of (1) comes from the realization that the same situation holds for the PE, which is written as follows:

$$\begin{aligned} &\lambda_1 \int dx \int dy\ \psi^\dagger(x)\,\psi(x)\,V(x-y)\,\psi^\dagger(y)\psi(y) \\ &\quad = (\lambda_1/L)\sum_{q>0} V_q(\rho_q^\dagger \rho_q + \rho_q \rho_q^\dagger) \\ &\quad = (\lambda_1/\pi)\sum_{q>0}(qV_q)(a_q^\dagger a_q + \tfrac{1}{2}), \end{aligned} \tag{10}$$

in which V_q is the Fourier transform of $V(x)$, presumed real. Combining the above, we obtain H in the form:

$$H = E_0 + \sum_{q>0}\omega_q a_q^\dagger a_q, \tag{11}$$

where E_0 is the sum of the ground state and renormalization energies,

$$E_0 = \sum_{k<k_F} k + (\lambda_1/2\pi)\sum_{q>0} qV_q, \tag{12}$$

and the renormalized normal-mode energies are

$$\omega_q = q(1 + \lambda_1 V_q/\pi). \tag{13}$$

We now turn to the strange solutions of (1). These are evidently confined to the case N = finite, in which the Dirac–Fermi sea is unfilled. Nevertheless, such a case is frequently considered in conjunction with a cutoff k_c below which no fermions are to be permitted.[4] The limit $k_c \to -\infty$

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is taken after the solution is obtained, under the assumption that many of the properties are cutoff independent.

The energy eigenstates of (1) are written as

$$\Psi = F(x_1,x_2,\cdots,x_N)\prod_{n=1}^{n}\psi^+(x_n)|0), \tag{14}$$

where $F(x_1,\cdots)$ satisfies the partial differential equation

$$-i\sum_{n=1}^{N}\frac{\partial F}{\partial x_n}+\lambda_1\sum_{n,m}V(x_n-x_m)\,F=EF(x_1,\ldots) \tag{15}$$

and E is the energy eigenvalue. F is recognized as the ordinary wavefunction in first quantization. The anticommutation relations applied to (14) limit us to solutions of (15) that are totally antisymmetric, i.e., that change sign under the interchange of any pair of coordinates x_n, x_m. It is these solutions that we now determine. The key observation here is that the KE operator, identical to the total momentum operator,

$$-i\sum_n\frac{\partial}{\partial x_n} \tag{16}$$

commutes with the PE operator

$$\lambda_1\sum_{n,m}V(x_n-x_m). \tag{17}$$

An additional simplification comes from the fact that the KE involves only first derivatives. If $G(x_1,\cdots)$ is defined as the symmetric free field ($\lambda_1=0$) solution of (15) belonging to energy E_1 and $F(x_1,\cdots)$ is the solution of (15) with energy E_2, then it is easily seen that $G(x_1,\cdots)F(x_1,\cdots)$ is also a solution of (15) belonging to energy eigenvalue $E=E_1+E_2$.

Taken together, these remarks determine what can be shown to be, in fact, the most general form of the solution; the argument goes as follows: the kinetic energy depends upon a single variable—the center of mass coordinate; the potential energy does not depend upon it at all. Thus, we may write the eigenfunction as

$$F=\exp\left\{i\sum x_j\left[E-\lambda_1\sum V(x_n-x_m)\right]\Big/N\right\}\cdot f(x_N-x_{N-1},\ldots,x_2-x_1). \tag{18}$$

Since the exponential phase is symmetric under permutation of particles, the function f of the relative coordinates will be antisymmetric. As it stands, this is an eigenstate for any value of energy; these states are the scattering states of our system. We now impose periodic boundary conditions on the system; this has the effect of fixing the phase so that $[E-\lambda_1\Sigma V(x_n-x_m)]\,L\equiv KL=\pi x$ (even integers) for N odd and πx (odd integers) for N even. This, however, can only hold if $\Sigma\,V(x_n-x_m)$ is constant, and thus the function f of the relative coordinates is required, by periodic boundary conditions, to be a product of delta functions fixing the relative separations $x_n-x_{n-1}=r_n$. We then arrive at our final form:

$$F=e^{iKx_1}\prod_{n=2}^{N}\delta(x_n-x_{n-1}-r_n) \tag{19}$$

for $x_1<x_2\cdots<x_N$. For any permutation on this ordering one introduces a factor $(-1)^P$ to satisfy the Pauli principle. Thus, as previously derived, the periodic boundary condition requires $KL=\pi\times$(even integer) for $N=$ odd, and $\pi\times$(odd integer) for $N=$ even. The r_n's are a set of arbitrary nearest neighbor separations, subject only to $d\equiv\Sigma r_n<L$, and can be used to compute x_n-x_m, which we may define as r_{nm}, constants of the motion. (Note: $r_{N1}=L-d$, by periodic boundary conditions.) The energy eigenvalue corresponding to the solution (18) is

$$E=K+\lambda_1\sum_{n,m}V(r_{nm}). \tag{20}$$

In a calculation of the partition function, the kinetic and potential energies contribute separately just as in classical physics; quantum mechanics seems to play a negligible role in discretizing K. The spectrum of energies (20) bears no discernible relation to (11)–(13).

These "strange" solutions may be viewed as the condensation of fermions into a "Wigner lattice" [the value of the r_{nm} which minimizes the total energy (20)].

The "strangeness" is compounded when it is seen that these solutions do *not* reduce to the expected determinantal wavefunctions,

$$F_0=(N!)^{-1/2}\det\|e^{ik_jx_n}\| \tag{21}$$

when $\lambda_1\to 0$. Such determinantal functions are only valid at precisely $\lambda_1=0$, where they can be constructed by taking a linear combination of F's belonging to different sets of $\{r_n\}$, which are degenerate only when the PE is extinguished. So the "intuitively obvious" eigenstates (21) of the noninteracting system are not even the natural limiting functions! This is a particularly clear example of the "tracks of the vanished dinosaurs" that Klauder[5] has remarked in various examples of field theory: The influence of interactions persists in the strange form of the solution even after the coupling constant vanishes.

Our second example concerns Luttinger's model[6] of a two-component field theory. This example is nontrivial in the sense that KE and PE do not now commute. Nevertheless, a set of "strange solutions" persists, differing qualitatively from Luttinger's own solutions[6] in the same way that the strange solutions found above differed from the determinantal functions. But this discrepancy may be academic, for *none* of these are physically acceptable; the physically acceptable solutions are obtained quite differently, by first filling the Fermi–Dirac sea and then examining the operators in the manner first prescribed by Mattis and Lieb.[1] The model Hamiltonian is now

$$\begin{aligned}H=&-i\int dx\,[\psi^+(x)\partial_x\psi(x)-\phi^+(x)\partial_x\phi(x)]\\&+\lambda_1\int dx\,[dx'\,\psi^+(x)\,\psi(x)\,V(x-x')\,\psi^+(x')\,\psi(x')]\\&+\lambda_1\int dy\int dy'\,\phi^+(y)\,\phi(y)\,V(y-y')\,\phi^+(y')\,\phi(y')\\&+2\lambda_2\int dx\int dy\,\psi^+(x)\,\psi(x)\,U(x-y)\,\phi^+(y)\,\phi(y).\end{aligned} \tag{22}$$

The exact eigenstates—found by first filling the Fermi–Dirac seas ($k<k_{F1}$ for the ψ particles, $k>k_{F2}$ for the ϕ particles) and then following the prescriptions of Eqs. (5b)–(13)—

now involve a complete set of q's (positive for the density fluctuations in the ψ particles, negative for ϕ 's). The relevant Hamiltonian is

$$H = E_0' = \sum_{q>0} q(1+\lambda_1 V_q/\pi)(a_q^+ a_q + a_{-q}^+ a_{-q}) + (\lambda_2/\pi)\sum_{q>0} U_q q(a_q^+ a_{-q}^+ + \text{H.c.}), \tag{23}$$

where

$$E_0' = \sum_{k<k_{F1}} k - \sum_{k>k_{F2}} k + (\lambda_1/\pi)\sum_{q>0} V_q q. \tag{24}$$

The final step is a Bogoliubov transformation to a new set of normal modes diagonalizing (23):

$$a_q = \cosh u_q b_q + \sinh u_q b_{-q}^+. \tag{25}$$

With

$$\tanh 2u_q = \frac{-\lambda_2 U_q/\pi}{1+\lambda_1 V_q/\pi} \tag{26}$$

we obtain

$$H = E_0'' + \sum_{\text{all}} \omega_q' b_q^+ b_q, \tag{27}$$

where

$$\omega_q' = |q|\,[(1+\lambda_1 V_q/\pi)^2 - (\lambda_2 U_q/\pi)^2]^{1/2} \tag{28}$$

and

$$E_0'' = E_0' + \sum_{q>0}(\omega_q' - \omega_q). \tag{29}$$

Turning next to finite N_1 and N_2, we find only solutions of the strange variety. First, write the eigenstates in the form

$$\Psi = F(x_1,\ldots,x_{N1};y_1,\ldots,y_{N2})\prod \psi^+(x_1)\cdots\phi^+(y_1)\cdots|0), \tag{30}$$

and study the eigenvalue equations for F

$$-i\sum \frac{\partial F}{\partial x_n} + i\sum \frac{\partial F}{\partial y_m} + \lambda_1 VF + 2\lambda_2 UF = EF, \tag{31}$$

where

$$V \equiv \sum_{n,m}[V(x_n - x_m) + V(y_n - y_m)],$$

$$U \equiv \sum_{n,m} U(x_n - y_m).$$

Borrowing from the previous procedure, we fix $x_2,\ldots,x_{N1}$ relative to x_1 and similarly for the y's:

$$x_2 = x_1 + r_2,\ x_3 = x_1 + r_2 + r_3,\cdots,$$

$$y_2 = y_1 + r_2',\ y_3 = y_1 + r_2' + r_3',\cdots,$$

where the r's are constants of the motion. Thus, V, which depends only on the r's, is itself a constant of the motion while U depends on the coordinates only through $x_1 - y_1$. We write this dependence explicitly as $U(x_1 - y_1)$.

The partial differential equation (31) reduces then to the simpler problem:

$$-i\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}\right)F + 2\lambda_2 U(x_1 - y_1)F = (E - \lambda_1 V)F. \tag{32}$$

With

$$W(z) \equiv \int^z dx\, U(x) \tag{33}$$

we obtain the most general solution:

$$F = e^{i(k_1 x_1 + k_2 y_1)} e^{i\lambda_2 W(x_1 - y_1)} \times \Pi\delta(x_n - x_{n-1} - r_n)\,\delta(y_m - y_{m-1} - r_m'). \tag{34}$$

Periodic boundary conditions on x yield the magnitude of k_1:

$$k_1 L - \lambda_2 W(L) = p_1 \pi, \tag{35}$$

where p_1 is an even/odd integer depending on whether N_1 is odd/even. A similar equation is constructed for k_2, and the energies are found to be

$$E = k_1 - k_2 + \lambda_1 V \tag{36}$$

independent of λ_2 except through the quantization condition (35). Note the interesting consequence that whenever λ_2 is increased such that $\lambda_2 W(L)$ increases by a multiple of 2π, there is no effect on the eigenstates save a relabeling. Again, the strange solutions appear to have no discernible physical interpretation. The more plausible states guessed by Luttinger[6] can be obtained *only if* $\lambda_1 \equiv 0$. Luttinger was, of course, careful to make this restriction in his original paper[6] as well as the simplifying assumption $W(L) = 0$. By lifting these restrictions, we have pointed out some of the difficulties that might not otherwise have been apparent.

The analysis can be generalized further, in two significant ways, which preserve the dichotomy between acceptable and strange solutions.

First, we can relax the requirements on the potential energy that it depend on differences $x_n - x_m$ or $x_n - y_m$ of the coordinates, and consider two-body forces that depend in an arbitrary way on the coordinates x_n and y_m. On the one hand, the Hamiltonian remains a quadratic form in Tomonaga operators regardless of the nature of the two-body forces, and can always be diagonalized by standard methods. On the other, the strange solutions can always be found because in the $(N_1 + N_2)$-dimensional coordinate space, the kinetic energy is effective only along one axis, viz., $\zeta_1 = (N_1 + N_2)^{-1/2}(x_1 + x_2 + \cdots - y_1 - y_2\cdots)$, whereas it commutes with all coordinates $\zeta_2, \zeta_3,\ldots,\zeta_{N1+N2}$ along the orthogonal axes. Therefore, the eigenvalue equation for F can always be transformed into an ordinary first-order, linear differential equation in the variable ζ_1 and solved exactly (albeit, with a physically unacceptable result!)

Second, we can introduce spin as a variable. The particles are then labeled according to their spin component (up or down) as well as their motion (right- or left-going). As long as the numbers of particles in each component are conserved, the models remain soluble in both second and first quantization, with the strange solutions found in the latter.

We have found no simple prescription that "heals" the strange solutions. The introduction of a cutoff does not restore a proper form to the eigenstates nor a proper set of dispersion relations to the excitations. Proceeding to the limit $N \to \infty$ does not help the situation.

In a future paper we plan to explore other facets of this interesting area in mathematical physics, and analyze models which are somewhat more complex and interesting than

the above, having applications in field theory as well as in condensed matter theory.

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